JOURNAL OF DIFFERENTIAL EQUATIONS **127**, 519–541 (1996)
ARTICLE NO. 0081

Nonautonomous Lotka–Volterra Systems, I

RAY REDHEFFER

Department of Mathematics, University of California, Los Angeles, California 90025

Received May 9, 1994; revised May 15, 1995

FOR HEDDY

This paper is concerned with the Lotka–Volterra system

$$E: \quad \dot{u}_i = u_i \left(a_i(t) - \sum_{j=1}^{n} b_{ij}(t) u_j \right), \qquad t > t_0; \qquad u_i(t_0) > 0$$

for $i = 1, ..., n$, where a_i and b_{ij} are continuous real-valued functions of the real variable t. Each u_i in E satisfies an equation of the form $\dot{u}_i = u_i \phi_i$, where $\phi_i(t)$ is continuous. Hence $u_i > 0$ holds trivially on the interval of existence of u. If u_i admits a bound on (t_0, T) independent of T, then u exists on (t_0, ∞). The equation is interpreted for $t \in \mathbb{R}$ by setting $t_0 = -\infty$ and replacing the condition $u(t_0) > 0$ by $u(t_1) > 0$ for some value t_1. Unless the hypothesis $t \in \mathbb{R}$ is mentioned explicitly, $t > t_0$ and $t_0 \in \mathbb{R}$.

We use the notation

$$a = \text{vector}(a_i) = a(t), \quad B = \text{matrix}(b_{ij}) = B(t), \quad u = \text{vector}(u_i) = u(t).$$

The letters c, d, $\bar{d}$ denote positive vectors of $\mathbb{R}^n$ partially ordered as follows:

$$c \leqslant d \quad \text{means} \quad c_i \leqslant d_i, \quad c < d \quad \text{means} \quad c_i < d_i, \qquad i = 1, ..., n.$$

Inequalities between vector-valued functions are interpreted accordingly; in particular, $\inf u > 0$ means $\inf u_i(t) > 0$ for $i = 1, ..., n$. For any real-valued function ϕ we define

$$\phi^-(t) = \min(\phi(t), 0), \quad \phi^+(t) = \max(\phi(t), 0). \tag{1}$$

If a condition involving i, j or t is stated without further explanation, it holds for $i, j = 1, ..., n$ and for $t > t_0$. To simplify the statement, it is assumed in Theorem 1 that

$$b_{ii} > 0, \qquad i = 1, 2, ..., n.$$

The extent to which this hypothesis can be weakened will be clear from the lemmas.

0022-0396/96 $18.00

1. Objectives

This paper is influenced more by an unpublished manuscript of Ahmad and Lazer [4] than by the published literature. The initial objective was merely to extend the A-L results to equations in which $b_{ij} \geqslant 0$ is no longer assumed. However it was soon seen that the proofs in [4] depend on a lemma that is true for continuous functions but not for the class of discontinuous functions for which it is needed. In the course of supplying new proofs, it was found that some of the results are correct as stated, while some require modification.

Under the hypothesis $a_i(t) > 0$, $b_{ii}(t) > 0$, $b_{ij}(t) \geqslant 0$ for $t \in \mathbb{R}$, Ahmad and Lazer introduce the inequalities

$$d_i = \sup_t \frac{a_i(t)}{b_{ii}(t)} < \infty, \qquad \inf_t \frac{a_i(t) - \sum_{j \neq i} b_{ij}(t)\, d_j}{b_{ii}(t)} > 0.$$

These form the starting-point of the present investigation. If the inf is denoted by $\bar{d}_i$, the inequalities become

$$a_i \leqslant b_{ii} d_i, \qquad a_i - \sum_{j \neq i} b_{ij} d_j \geqslant b_{ii} \bar{d}_i. \tag{2}$$

We will replace (2) the following, which reduces to (2) when $b_{ij} \geqslant 0$:

$$b_{ii} \bar{d}_i + \sum_{j \neq i} b_{ij}^+ \, d_j \leqslant a_i \leqslant b_{ii} d_i + \sum_{j \neq i} b_{ij}^- \, d_j. \tag{a}$$

If the left member of (a) is subtracted from the right, the result is a condition of *row diagonal dominance*,

$$b_{ii} d_i - \sum_{j \neq i} |b_{ij}| \, d_j \geqslant b_{ii} \bar{d}_i.$$

This shows that $|b_{ij}|/b_{ii}$ is bounded if $b_{ii} > 0$, a fact needed later. More important for our purposes is a condition of *column diagonal dominance*,

$$c_j b_{jj} - \sum_{i \neq j} c_i \, |b_{ij}| \geqslant c_j \, \delta(t) \geqslant 0, \tag{b}$$

where $\delta(t)$ is continuous. Finally, we define $\gamma(t) = \min_i b_{ii}(t)$ as in [4] and introduce the three conditions

$$\int_{t_0}^{\infty} \gamma(s)\, ds = \infty, \qquad \int_{-\infty}^{t_0} \delta(s)\, ds = \infty, \qquad \int_{t_0}^{\infty} \delta(s)\, ds = \infty. \tag{cde}$$

THEOREM 1. *Let E and* (abcde) *be as described above, and suppose $b_{ii} > 0$.*

(i) *If* (a) *holds and $\bar{d} \leqslant u(t_0) \leqslant d$, then u exists for $t > t_0$ and satisfies $\bar{d} \leqslant u(t) \leqslant d$.*

(ii) *If* (a) *holds on $\mathbb{R}$, there exists a solution u^* of E on $\mathbb{R}$ satisfying $\bar{d} \leqslant u^* \leqslant d$.*

(iii) *If* (ac) *hold then u exists for $t > t_0$ and* $0 < \inf u(t) \leqslant \sup u(t) < \infty$.

(iv) *If* (ac) *hold and E has a solution u^* such that* $\lim_{t \to \infty} u^*(t) = q$ *exists, then every solution satisfies* $\lim_{t \to \infty} u(t) = q$.

(v) *If* (abd) *hold on $\mathbb{R}$, then u^* in* (i) *is the only solution on $\mathbb{R}$ that is bounded away from* 0 *and* ∞.

(vi) *If* (abce) *hold then* $\lim_{t \to \infty} |\bar{u}(t) - u(t)| = 0$ *holds for every pair of solutions u, $\bar{u}$.*

(vii) *If* (ab) *hold on $\mathbb{R}$ and a and B are periodic with the same period, then E has one and only one periodic solution u^*.*

(viii) *If* (ab) *hold on $\mathbb{R}$ with δ constant and a and B are almost periodic, then E has one and only one almost periodic solution u^*.*

(ix) *If B is constant,* (i)–(viii) *remain true under the hypothesis* (a) *alone.*

Note that (i)–(iv) do not involve (b) and (v)–(viii) do. Before turning to the proof we make some comments.

Remark 1. If $\bar{d} = d$ then (a) implies $b_{ij} = 0$ for $i \neq j$ and the conclusion (i) implies $u = d$. Since $u = d$ is actually a solution, the result remains valid even in this extreme case.

Remark 2. The multipliers d_i can be introduced by a change of scale, thus reducing the general case to the case $d_i = 1$. Let u denote a solution of E, and define v by $u_i(t) = d_i v_i(t)$. Then v satisfies the same equation with (b_{ij}) replaced by $(b_{ij} d_j)$, namely

$$\dot{v}_i = v_i(a_i(t) - \sum b_{ij}(t)\, d_j v_j).$$

Remark 3. The solutions of E are called *weakly permanent* if they all have the property described in (iii), and *permanent* if there are positive constants α, β, independent of the initial conditions, such that every solution satisfies

$$\alpha \leqslant \liminf_{t \to \infty} u(t) \leqslant \limsup_{t \to \infty} u(t) \leqslant \beta.$$

Thus (iii) asserts that the solutions are weakly permanent and (iv) and (vi) that they are permanent.

Remark 4. Under the hypothesis (2) with $a_i > 0$, $b_{ij} \geqslant 0$, $b_{ii} > 0$, Ahmad and Lazer assert the existence of a solution u^* bounded away from 0 and ∞ on $\mathbb{R}$. Theorem 1 (ii) shows that this important part of their theorem is correct.

Remark 5. For (v)–(viii) Ahmad and Lazer use conditions like (de) with γ instead of δ, and this leads to one of the most subtle and interesting questions about Lotka–Volterra systems that has turned up in quite a while. The crucial assertion is (vi). According to (iv), the hypothesis (ac) implies (vi) if some solution has a limit, and by (ix), the same holds if the coefficients are constant. But the question whether (ac) always implies (vi) has been open for several years. We will show that the answer is negative even when $b_{ij} \geqslant 0$, so introduction of (b) as a separate hypothesis is really necessary. The proofs of this and of (iv) are given in Part II.

2. Estimates Involving Initial Conditions

We will prove the following:

Lemma 1. *Suppose there is a vector $d > 0$ such that $a < \gamma$ where*

$$\gamma_i = b_{ii} d_i + \sum_{j \neq i} b_{ij}^- d_j.$$

Then $u(t_0) < d \Rightarrow u(t) < d$. If $\gamma > 0$ the same holds with each of the three inequalities $<$ replaced by $\leqslant$.

Proof. Suppose the inequality $u(t) < d$ fails. Then there is a smallest value $t = t^* > t_0$ and an index i such that $u_i = d_i$ and $u_j \leqslant d_j$ at $t = t^*$. Since $\dot{u}_i(t^*) \geqslant 0$, Equation E at t^* gives

$$a_i \geqslant b_{ii} u_i + \sum_{j \neq i} b_{ij} u_j \geqslant b_{ii} u_i + \sum_{j \neq i} b_{ij}^- u_j \geqslant b_{ii} d_i + \sum_{j \neq i} b_{ij}^- d_j,$$

which is a contradiction. To deal with weak inequality when $\gamma > 0$, replace d by θd, where $\theta > 1$. The theorem with $<$ gives $u < \theta d$, and the conclusion follows when $\theta \to 1$.

The same conclusion, with the same proof, holds if $d = d(t)$ is a positive function satisfying $\dot{d} \geqslant 0$. The point is worth mentioning, because it brings the subject into contact with the theory of quasimonotone operators. Indeed, if $\dot{d} \geqslant 0$ then the hypothesis of Lemma 1 and E give respectively

$$\frac{\dot{d}_i}{d_i} > a_i - b_{ii} d_i - \sum_{j \neq i} b_{ij}^- d_j, \qquad \frac{\dot{u}_i}{u_i} \leqslant a_i - b_{ii} u_i - \sum_{j \neq i} b_{ij}^- u_j.$$

Thus the introduction of ϕ^+ and ϕ^- for suitable functions ϕ has converted the nonmonotone equation E into an inequality of quasimonotone structure. A similar extension can be given for other results involving multipliers. In particular, the following Lemma 2 holds if $\bar{d}=\bar{d}(t)$ is decreasing and $d=d(t)$ increasing, both being positive and differentiable:

LEMMA 2. *Suppose $u_j \leqslant d_j$ and that $\bar{d}$ is a positive vector satisfying*

$$\bar{d}<d, \qquad b_{ii}\bar{d}_i+\sum_{j\neq i} b_{ij}^+ d_j<a_i.$$

Then $\bar{d}<u(t_0) \Rightarrow \bar{d}<u(t)$. If $b_{ii}>0$, the same holds with $<$ replaced by $\leqslant$ in each of its four occurrences.

Proof. If the inequality $u(t)>\bar{d}$ fails, there is a value $t^*>t_0$ and an index i such that $u_i=\bar{d}_i$ and $\dot{u}_i \leqslant 0$ at t^*. Since $u_j \leqslant d_j$, this gives a contradiction. The case $\leqslant$ is dealt with by applying the case $<$ with $\bar{d}$ replaced by $\theta\bar{d}$ and letting $\theta \to 1-$.

Proof of (i). If (a) holds and $b_{ii}>0$, Lemmas 1 and 2 both apply in the $\leqslant$ form. This gives Theorem 1 (i).

Proof of (ii). Let I_1 denote the interval $|t| \leqslant 1$ and let $m>1$ be an integer. We denote by $u=v_m$ the solution of E satisfying

$$v_m(-m)=\tfrac{1}{2}(\bar{d}+d).$$

Theorem 1 (i) gives $\bar{d} \leqslant v_m(t) \leqslant d$ on I_1. These uniform bounds for v_m give a uniform bound for $\dot{v}_m$ on I_1, since the coefficients in E are continuous. Hence the family $\{v_m\}$ is uniformly equicontinuous, so there is a subsequence $\{v_{m,1}\}$ that converges to a solution of E on the interior of I_1.

More generally, let I_j denote the interval $|t| \leqslant j$ where j is any positive integer. The argument used above shows that $\{v_{m,1}\}$ has a subsequence $\{v_{m,2}\}$ that converges to a solution on the interior of I_2. From this we can select a subsequence $\{v_{m,3}\}$ that converges on the interior of I_3, and so on. The diagonal sequence $\{v_{m,m}\}$ converges on the interior of I_j for every j and yields the solution u^*.

3. BEHAVIOR IN THE REMOTE FUTURE

A real-valued function $m(t)$ defined on an open interval I is called *Lipschitzian* if, for each finite closed subinterval $J \subset I$ there is a constant $k=k(J)$ such that

$$|m(t_1)-m(t_2)| \leqslant k\,|t_1-t_2|, \qquad t_1, t_2 \in J.$$

We do not require a uniform constant k on I. For example, the solutions of E are Lipschitzian, since the coefficients are continuous. A Lipschitzian function m is locally absolutely continuous, so $\dot{m}$ exists almost everywhere and m is the integral of its derivative.

LEMMA 3. *Let $\theta > 0$, $T > t_0$ and let g be a continuous function satisfying*

$$g(t) \geqslant 0, \quad \int_T^\infty g(t)\, dt = \infty.$$

Suppose m is a Lipschitzian function such that $\dot{m}(t) \leqslant g(t)\, m(t)(\theta - m(t))$ at all points $t > T$ where $\dot{m}(t)$ exists and $m(t) > \theta$. Then $\limsup_{t \to \infty} m(t) \leqslant \theta$.

Proof. We distinguish two cases.

Case 1. If $m(t_1) \leqslant \theta$ at some $t_1 > T$, then $m(t) \leqslant \theta$ for all $t \geqslant t_1$. Otherwise $m(t_3) = \bar{\theta} > \theta$ at some value $t_3 > t_1$. We take t_3 to be the first point beyond t_1 where $m(t) = \bar{\theta}$. Starting at t_3, we diminish t until we first reach a value t_2 at which $m(t_2) = \theta$. Then $\theta < m(t) < \bar{\theta}$ on (t_2, t_3), so $\dot{m} \leqslant gm(\theta - m)$ holds at all points of this interval in which $\dot{m}$ exists. At any such point $\dot{m} \leqslant 0$, so integration gives $m(t_3) \leqslant m(t_2)$, a contradiction.

Case 2. If $m(t) > \theta$ for all $t > T$, then $\dot{m} \leqslant gm(\theta - m)$ holds at every point where $\dot{m}$ exists. We linearize by setting $y = 1/m$. The result is

$$\theta y(t) \geqslant 1 + c_0 e^{-\theta \int_T^t g(s)}\, ds,$$

where c_0 is a constant whose value need not concern us. This gives

$$\lim_{t \to \infty} \theta y(t) \geqslant 1, \qquad \text{hence} \qquad \lim_{t \to \infty} m(t) \leqslant \theta.$$

LEMMA 4. *Let $m(t) = \max_i m_i(t)$ where each $m_i(t)$ is differentiable on an open interval I. Then at any point of I where $\dot{m}$ exists there is an index $i = i(t)$ such that $m(t) = m_i(t)$ and $\dot{m}(t) = \dot{m}_i(t)$.*

Proof. Let $t > t_0$ be a point where $\dot{m}(t)$ exists, and let $m(t) = m_p(t)$. Then for any $h > 0$ there is an index $j = j(h)$ such that

$$m(t+h) - m(t) = m_j(t+h) - m_p(t).$$

As $h \to 0$ through a null sequence $\{h\}$, some index $j(h)$ must repeat infinitely often. We denote this index by i. Continuity gives $m_p(t) = m_i(t)$, so

$$\frac{m(t+h) - m(t)}{h} = \frac{m_i(t+h) - m_i(t)}{h}$$

on the null sequence $\{h\}$. Hence $\dot{m}(t) = \dot{m}_i(t)$.

LEMMA 5. *Suppose there exists a positive vector d such that*

$$d_i b_{ii} + \sum_{j \neq i} b_{ij}^{-} d_j = \gamma_i, \tag{3}$$

where the function $\gamma(t) = \min_i \gamma_i(t)$ *satisfies*

$$\gamma(t) > 0, \quad \int_{t_0}^{\infty} \gamma(t)\, dt = \infty.$$

Suppose also that $\limsup_{t \to \infty} a_i(t)/\gamma_i(t) \leqslant 1$. *Then* $\limsup_{t \to \infty} u(t) \leqslant d$.

Proof. Since each u_i is Lipschitzian, the function

$$m(t) = \max_i \frac{u_i(t)}{d_i}$$

is also. Let $t > t_0$ be a point where $\dot{m}(t)$ exists. Using the index i given by Lemma 4, and suppressing the variable t for brevity, we have at t

$$\dot{m} = \dot{m}_i = m\left(a_i - b_{ii} d_i m - \sum_{j \neq i} b_{ij} u_j\right).$$

If we change b_{ij} to b_{ij}^{-}, this changes the second $=$ to $\leqslant$. The fact that $u_j \leqslant d_j m$ then yields

$$\dot{m} \leqslant m\left(a_i - b_{ii} d_i m - \sum_{j \neq i} b_{ij}^{-} d_j m\right) = m(a_i - \gamma_i m).$$

Given $\theta > 1$, let T be such that $a_i(t) \leqslant \theta \gamma_i(t)$ for $t \geqslant T$. Then

$$\dot{m} \leqslant \gamma_i m(\theta - m)$$

at all points $t > T$ where $\dot{m}$ exists. If $m \geqslant \theta$ (but only then) we get

$$\dot{m} \leqslant \gamma m(\theta - m) \qquad \text{where} \quad \gamma(t) = \min_i \gamma_i(t), \qquad t \geqslant T.$$

Divergence of the integral in Lemma 5 implies $\limsup_{t \to \infty} m(t) \leqslant \theta$ by Lemma 3. The conclusion follows when $\theta \to 1$.

The following remark illustrates the added simplicity that is to be expected when $b_{ij} \geqslant 0$, as in [4]. Note also that (4) is much weaker than the corresponding hypothesis in Lemma 5:

Remark 6. Suppose $b_{ij} \geqslant 0$, $b_{ii} > 0$ and $\limsup_{t\to\infty} a_i(t)/b_{ii}(t) \leqslant d_i$. Suppose further that

$$\int_{t_0}^{\infty} b_{ii}(t)\, dt = \infty. \tag{4}$$

Then $\limsup_{t\to\infty} u(t) \leqslant d$. This follows from the fact that, if $\theta > 1$, then for large t

$$\dot{u}_i \leqslant u_i(a_i - b_{ii}u_i) \leqslant b_{ii}u_i(\theta d_i - u_i).$$

LEMMA 6. *Suppose that* $b_{ii} \geqslant 0$, $\limsup_{t\to\infty} u(t) \leqslant d$, *and that there is a positive constant* ρ *such that*

$$\liminf_{t\to\infty} \frac{a_i(t)}{\rho b_{ii}(t) + \sum_{j\neq i} b_{ij}^+ d_j} > 1. \tag{5}$$

Then $\liminf_{t\to\infty} u(t) > 0$. *If* $b_{ii} > 0$ *and* b_{ij}/b_{ii} *is bounded above, the same strict inequality in the conclusion follows when* $>$ *in* (5) *is replaced* by $\geqslant$.

Proof. We prove the first statement and reduce the second to it.

Case 1. Given (5), choose $\theta > 1$ but so close to 1 that the following holds for some T:

$$a_i \geqslant \theta\rho b_{ii} + \theta \sum_{j\neq i} b_{ij}^+ d_j, \qquad t \geqslant T.$$

By increasing T we can suppose also that $u_i \leqslant \theta d_i$ for $t \geqslant T$. Hence $t \geqslant T$ implies

$$\dot{u}_i \geqslant u_i \left(a_i - b_{ii}u_i - \sum_{j\neq i} b_{ij}^+ u_j \right) \geqslant b_{ii}u_i(\theta\rho - u_i).$$

If $u_i \leqslant \rho$ this makes $\dot{u}_i \geqslant 0$, so $u_i \geqslant \min(u_i(T), \rho)$, $t \geqslant T$.

Case 2. If equality holds in (5) with $b_{ii} > 0$ and b_{ij}/b_{ii} bounded above, choose a constant μ such that

$$\sum_{j\neq i} b_{ij}^+ d_j \leqslant \mu b_{ii}$$

and let $0 < \bar{\rho} < \rho$. Then

$$\liminf_{t\to\infty} \frac{a_i(t)}{\bar{\rho} b_{ii}(t) + \sum_{j\neq i} b_{ij}^+ d_j} \geqslant \frac{\rho + \mu}{\bar{\rho} + \mu} > 1,$$

which reduces to Case 1.

Proof of (iii). Under the hypothesis of Theorem 1 (iii) we have

$$b_{ii}\bar{d}_i + \sum_{j \neq i} b_{ij}^{+} d_j \leqslant a_i \leqslant d_i b_{ii} + \sum_{j \neq i} b_{ij}^{-} d_j = \gamma_i$$

with γ_i as in Lemma 5. Clearly $\gamma_i \geqslant b_{ii}\bar{d}_i$ and $a_i/\gamma_i \leqslant 1$. Since the b_{ii} admit a common minorant with divergent integral the same is true of the γ_i, so Lemma 5 gives $\limsup_{t \to \infty} u(t) \leqslant d$. Half of Theorem 1 (iii) follows as a special case. In Section 1 we mentioned that b_{ij}/b_{ii} is bounded above when (a) holds. Hence Lemma 6 applies with $\leqslant$ and gives the other half.

4. Comparison of Two Solutions

In Section 3 we used the fact that certain functions are Lipschitzian to deal with differential inequalities that do not hold everywhere. A different approach, not depending on the theory of absolute continuity and Lebesgue integration, is given next. The alternative approach was used without proof in an earlier version of this paper and is now justified in response to a request by the Referee:

Lemma 7. *Let $m(t) = \max_i m_i(t)$ where each $m_i(t)$ is differentiable on an open interval I except in a countable subset. Then $m(t)$ is also differentiable on I except in a countable subset.*

In our application each m_i will be differentiable throughout I. Lemma 7 is stated in a stronger form than needed so we can use induction.

Proof. Taking $i \in \{1, 2\}$, let $m = \max(m_1, m_2)$ and let S denote the countable subset in which $\dot{m}_1$ and $\dot{m}_2$ do not both exist. If $\dot{m}(t_1)$ fails to exist at some point $t_1 \in I$, then we must have either $t_1 \in S$ or all three of the conditions

$$t_1 \in I - S, \qquad m_1(t_1) = m_2(t_1), \qquad \dot{m}_1(t_1) \neq \dot{m}_2(t_1).$$

These show that $m_1(t) \neq m_2(t)$ holds in a deleted neighborhood of t_1, so the points t_1 of this second type are isolated. Therefore the set of all such points is countable, and its union with S is also countable. Lemma 7 follows by induction.

Given a vector $c > 0$, we define a corresponding norm by

$$|x|_c = c_1\,|x_1| + c_2\,|x_2| + \cdots + c_n\,|x_n|, \qquad x \in \mathbb{R}^n. \tag{6}$$

LEMMA 8. *Let u, v be two solutions of E satisfying $\bar{\sigma} \leqslant u, v \leqslant \sigma$ for positive constants $\bar{\sigma}, \sigma$. Suppose there is a vector $c > 0$ such that*

$$c_j b_{jj} - \sum_{i \neq j} c_i |b_{ij}| \geqslant c_j \delta(t) \geqslant 0 \tag{b}$$

where $\delta(t)$ is continuous. Then

$$\bar{\sigma}\, |u(t) - v(t)|_c \leqslant \sigma\, |u(t_0) - v(t_0)|_c\, e^{-\bar{\sigma} \int_{t_0}^{t} \delta(s)\, ds}, \qquad t \geqslant t_0.$$

Proof. We use the function

$$V_c(t) = \sum_{i=1}^{n} c_i\, |\log u_i(t) - \log v_i(t)|,$$

which was introduced by Gopalsamy [8, 9] with $c_i = 1$; the extension to general c requires no new ideas. On any interval where $\operatorname{sgn}(u_i - v_i)$ is constant

$$\begin{aligned} \dot{V}_c &= \sum_i c_i \left(\frac{\dot{u}_i}{u_i} - \frac{\dot{v}_i}{v_i} \right) \operatorname{sgn}(u_i - v_i) \\ &= \sum_i \left(-c_i b_{ii}\, |u_i - v_i| - \sum_{j \neq i} c_i b_{ij} (u_j - v_j) \operatorname{sgn}(u_i - v_i) \right) \\ &\leqslant \sum_i \left(-c_i b_{ii}\, |u_i - v_i| + \sum_{j \neq i} |c_i b_{ij}|\, |u_j - v_j| \right). \end{aligned}$$

With $\xi_i = |u_i - v_i|$, the result for $n = 3$ is

$$\begin{aligned} \dot{V}_c \leqslant {} & -c_1 b_{11} \xi_1 + c_1 |b_{12}|\, \xi_2 + c_1 |b_{13}|\, \xi_3 \\ & + c_2 |b_{21}|\, \xi_1 - c_2 b_{22} \xi_2 + c_2 |b_{23}|\, \xi_3 \\ & + c_3 |b_{31}|\, \xi_1 + c_3 |b_{32}|\, \xi_2 - c_3 b_{33} \xi_3. \end{aligned}$$

Grouping terms by columns and using (b), we get

$$\dot{V}_c \leqslant -\delta\, |u - v|_c.$$

Clearly the result holds for arbitrary n. We refer to it as *Gopalsamy's inequality*, since the case $c_i = 1$ is due to him and the general case requires only trivial modification. By Lemma 7 Gopalsamy's inequality fails at most on a countable set, so it can be treated as if it holds everywhere [25]; for an expository account of this and related topics see [16].

By the mean value theorem, $\alpha, \beta > 0$ implies

$$\xi\, |\log \alpha - \log \beta| = |\alpha - \beta|$$

where ξ is between α and β. With $\alpha = u_i$ and $\beta = v_i$ this gives

$$\bar{\sigma} V_c \leqslant |u - v|_c \leqslant \sigma V_c, \tag{7}$$

so Gopalsamy's inequality leads to

$$\dot{V}_c \leqslant -\delta\, |u - v|_c \leqslant -\bar{\sigma}\, \delta V_c.$$

This gives an estimate for V_c that yields Lemma 8 by way of (7). It is only in this last step that we use the hypothesis $\delta \geqslant 0$. With $\bar{\sigma}\delta$ replaced by $\min(\sigma\delta, \bar{\sigma}\delta)$, we could allow δ to change sign.

Proof of (v). Let u and v be any solutions of E for $t \leqslant 0$ that are bounded away from 0 and ∞. Then (bd) implies $u = v$. This follows from Lemma 8 when we fix t and let $t_0 \to -\infty$. Theorem 1 (v) is an immediate consequence.

Remark 7. If (abd) hold, the technique used to prove (v) gives an alternative approach to (ii) by Cauchy sequences. Let $u = u_m$ be the solution satisfying $u(-m) = (c + d)/2$ for any positive integer m. Theorem 1 (i) gives a condition of the form $\bar{\sigma} \leqslant u \leqslant \sigma$ for $t \geqslant -m$ where the bounds $\bar{\sigma}, \sigma$ are independent of m. If (bd) hold, Lemma 7 together with (bd) shows that the family of solutions so obtained is a Cauchy sequence on any fixed interval (t_1, t_2) with respect to the norm $|\cdot|_c$, hence also with respect to the Euclidean norm. Therefore the family has a convergent subsequence. Since E is uniformly Lipschitzian on this interval, the limiting function is a solution. The uniqueness established in (v) allows us to expand the interval and get a solution for all t.

Proof of (vi). Suppose u and v are solutions of E for $t > t_0$ that are bounded away from 0 and ∞. Then (be) implies $\lim_{t \to \infty} (u(t) - v(t)) = 0$. This follows from Lemma 8 with t_0 fixed and $t \to \infty$. If (ac) hold the condition of boundedness follows from (iii). Since Theorem 1 (vi) assumes (abce), this completes the proof.

Proof of (vii). In the periodic case δ has a positive lower bound, hence can be replaced by a constant. Thus (b) implies (de). If a and B have period τ, the function $u^*(t + \tau)$ in (ii) satisfies the same equation as $u^*(t)$. Hence $u^*(t + \tau) = u^*(t)$ by (v). Note that τ need not be the smallest period of u^*; in fact, u^* could be constant.

Extending Lemma 8, we now consider two systems

$$E\colon\quad \frac{\dot{u}_i}{u_i} = a_i(t) - \sum_{j=1}^{n} b_{ij}(t)\, u_j, \qquad \tilde{E}\colon\quad \frac{\dot{v}_i}{v_i} = \tilde{a}_i(t) - \sum_{j=1}^{n} \tilde{b}_{ij}(t)\, v_j$$

with continuous coefficients satisfying

$$b_{ii}(t)+\sum_{j\neq i} b_{ij}^{-}(t)\geqslant 0, \qquad \tilde{b}_{ii}(t)+\sum_{j\neq i} \tilde{b}_{ij}^{-}(t)\geqslant 0.$$

Corresponding to the norm (6) we introduce a matrix norm for matrices $h=(h_{ij})\in\mathbb{R}^{n\times n}$, namely

$$|h|_c=\max_j(c_1\,|h_{1j}|+c_2\,|h_{2j}|+\cdots+c_n\,|h_{nj}|). \tag{8}$$

LEMMA 9. *Let u, v satisfy E, $\tilde{E}$ respectively and suppose $\bar{\sigma}\leqslant u$, $v\leqslant\sigma$ for positive constants $\bar{\sigma}$, σ. Let c and $\delta(t)$ be as in Lemma* 8. *Then*

$$\bar{\sigma}\,|u(t)-v(t)|_c\leqslant\sigma\,|u(t_0)-v(t_0)|_c\,e^{-\bar{\sigma}\Delta(t)}+\sigma\bar{\sigma}\bar{\eta}(t),$$

where $\eta(t)=|a(t)-\tilde{a}(t)|_c+|b(t)-\tilde{b}(t)|_c\sup v$ and

$$\Delta(t)=\int_{t_0}^{t}\delta(r)\,dr, \qquad \bar{\eta}(t)=e^{-\bar{\sigma}\Delta(t)}\int_{t_0}^{t}e^{\bar{\sigma}\Delta(s)}\eta(s)\,ds.$$

Proof. Set $\tilde{a}_i=a_i+h_i$, $\tilde{b}_{ij}=b_{ij}+h_{ij}$ and repeat the argument leading to Lemma 8. At the first step we get

$$\frac{\dot{u}_i}{u_i}-\frac{\dot{v}_i}{v_i}=(\text{old value})-\left(h_i+\sum_j h_{ij}v_j\right)\operatorname{sgn}(u_i-v_i),$$

where the “old value” is the one in the proof of Lemma 8. The error due to the extra terms h_i and h_{ij} leads to

$$\dot{V}_c\leqslant-|u-v|_c\,\delta+\eta\leqslant-\bar{\sigma}\,\delta V_c+\eta.$$

This gives an estimate for V_c that yields Lemma 9 by way of (7).

Proof of (viii). Let $\varepsilon>0$, and let τ be a translation number belonging to ε for all of the functions a_i and b_{ij}. (That a relatively dense set of such translation numbers exists was proved by Harald Bohr, as a lemma for showing that a finite sum of almost periodic functions is almost periodic. See [7, 33].) We will apply Lemma 9 with $u=u^*$, where u^* is the solution on $\mathbb{R}$ given by Theorem 1(i), and with $v(t)=u^*(t+\tau)$. Clearly v satisfies an equation like E with coefficients

$$a_i(t+\tau), \qquad b_{ij}(t+\tau).$$

Also v is bounded on $\mathbb{R}$ since u^* is. Hence

$$\eta(t)\leqslant\varepsilon(1+\sup u^*)\sum_{i=1}^{n}c_i.$$

If we choose t fixed and let $t_0 \to -\infty$ the result is

$$|u(t+\tau)-u(t)| = |v(t)-u(t)| \leqslant (\text{const})\,\varepsilon.$$

Hence τ is a translation number for u^* belonging to $\kappa\varepsilon$ for some constant κ. Uniqueness follows from (v).

The first introduction of almost periodic functions into the theory of differential equations may be due to Muckenhoupt; see [14, 33]. An early reference for systems of the type considered here is Amerio [6].

If $\eta \leqslant \kappa\delta$ where κ is a positive constant, $\bar{\sigma}\bar{\eta} \leqslant \kappa$, so Lemma 9 implies

$$\bar{\sigma}\,|u(t)-v(t)|_c \leqslant \sigma\kappa + \sigma\,|u(t_0)-v(t_0)|_c\, e^{-\bar{\sigma}\int_{t_0}^{t}\delta(s)\,ds}.$$

This leads to the following assertions of global asymptotic stability:

Remark 8. Under the hypothesis of Lemma 9, suppose that $\eta \leqslant \kappa\delta$ where κ is a positive constant. Then

$$\bar{\sigma}\,|u(t)-v(t)|_c \leqslant \sigma\kappa + \sigma\,|u(t_0)-v(t_0)|_c, \qquad t \geqslant t_0.$$

If in addition (e) holds, then

$$\bar{\sigma}\limsup_{t\to\infty}|u(t)-v(t)|_c \leqslant \sigma\limsup_{t\to\infty}\frac{\eta(t)}{\delta(t)},$$

and hence $\eta = o(\delta)$ implies $\lim_{t\to\infty}|u(t)-v(t)| = 0$.

5. Row and Column Dominance

This section reviews some known algebraic results on constant real matrices $B=(b_{ij}) \in \mathbb{R}^{n\times n}$. Some of the proofs seem to me simpler than those given hitherto; see [10, 11, 13, 15]. Nevertheless the discussion should be regarded as expository.

The matrix B is called *column diagonally dominant* or *row diagonally dominant* if

$$b_{jj} - \sum_{i\neq j}|b_{ij}| > 0 \qquad \text{or} \qquad b_{ii} - \sum_{j\neq i}|b_{ij}| > 0 \tag{9}$$

respectively. By a *multiplier* we mean a diagonal matrix with positive diagonal elements. The conditions (9) are far from being equivalent, but, as is well known, corresponding conditions with multipliers are equivalent. This matter is discussed here because of its general relevance to the subject of this paper and its particular relevance to Theorem 1 (ix).

The positive vectors c, d introduced above are correlated with the multipliers

$$C = \operatorname{diag} c_i, \qquad D = \operatorname{diag} d_i.$$

We denote by M_c the class of matrices $B = (b_{ij})$ such that CB is column diagonally dominant for some multiplier C, and by M_r the class of matrices B such that BD is row diagonally dominant for some multiplier D. For CB and BD the inequalities (9) change respectively to

$$c_j b_{jj} - \sum_{i \neq j} |c_i b_{ij}| > 0, \qquad b_{ii} d_i - \sum_{j \neq i} |b_{ij} d_j| > 0.$$

The *associated matrix* $\hat{B}$ is defined as usual by

$$\hat{b}_{ii} = b_{ii}, \qquad \hat{b}_{ij} = -|b_{ij}|, \qquad j \neq i.$$

Clearly B is diagonally dominant in either sense (row or column) if and only if $\hat{B}$ is diagonally dominant in the same sense.

LEMMA 10. *$M_r = M_c$, and B belongs to either of these classes if and only if* (i) *the diagonal elements of B are all positive*, (ii) *the matrix $\hat{B}$ is invertible, and* (iii) *the elements of $\hat{B}^{-1}$ are nonnegati*ve.

Since (iii) can be checked with ease, Lemma 10 gives a practical decision procedure.

Proof. The necessity of (i) is obvious from the definitions. For (ii) and (iii) we will use the matrix norm

$$|B| = \max_i (|b_{i1}| + |b_{i2}| + \cdots + |b_{in}|).$$

If $A = (a_{ij})$ and $B = (b_{ij})$ are real square matrices of the same size, then

$$|A + B| \leqslant |A| + |B|, \qquad |AB| \leqslant |A|\,|B|.$$

The first of these inequalities is evident and the second follows from

$$\max_i \sum_j \left| \sum_m a_{im} b_{mj} \right| \leqslant \max_i \sum_m |a_{im}| \sum_j |b_{mj}| \leqslant \max_i \sum_m |a_{im}|\,|B|.$$

Let $B \in M_r$. We find a multiplier D such that $\hat{B}D$ is row diagonally dominant, and then a multiplier C such that $C\hat{B}D$ has all diagonal elements 1. The left-hand factors c_i divide out of the inequalities that describe the row diagonal dominance of $\hat{B}D$, so $C\hat{B}D$ is still row diagonally dominant. Hence

$$C\hat{B}D = I - P$$

where $|P| < 1$ and P has nonnegative elements. Therefore

$$(C\hat{B}D)^{-1} = I + P + P^2 + \cdots$$

has nonnegative elements and, solving for $\hat{B}^{-1}$, we see that $\hat{B}^{-1}$ exists and has the same property.

Conversely, if (i), (ii) and (iii) hold, we can pick any positive vector y and solve $\hat{B}x = y$ for x. The solution $x = \hat{B}^{-1}y$ has nonnegative elements. In fact they are positive, since $x_i = 0$ implies that $\hat{B}^{-1}$ has only zeros in its ith row. If $D = \operatorname{diag} x_i$ then BD is row diagonally dominant, so the conditions (i), (ii), (iii) together imply $B \in M_r$. The fact that $M_c = M_r$ now follows from

$$B \in M_r \Leftrightarrow B^{\mathrm{T}} \in M_c$$

and from the fact that (i), (ii), (iii) hold for B^{T} if they hold for B.

Proof of (ix). If B is constant, (c) follows from $b_{ii} > 0$. Also (a) implies

$$d_i b_{ii} - \sum_{j \neq i} |b_{ij}|\, d_j \geqslant \varepsilon_1 d_i b_{ii}$$

where ε_1 is a positive constant. If we set $\tilde{b}_{ii} = (1 - \varepsilon_1)\, b_{ii}$ and $\tilde{b}_{ij} = b_{ij}$ for $j \neq i$, then $\tilde{B} \in M_r$. Lemma 10 gives $\tilde{B} \in M_c$, so there are positive constants c_i such that

$$c_j \tilde{b}_{jj} - \sum_{i \neq j} |c_i \tilde{b}_{ij}| > 0, \qquad j = 1, ..., n.$$

It follows that

$$c_j b_{jj} - \sum_{i \neq j} |c_i b_{ij}| > \varepsilon c_j b_{jj}$$

for a positive constant ε, so (bde) hold. This gives (ix) and completes the proof of Theorem 1.

The reason for assuming B constant in Theorem 1 (ix) is to ensure that the multipliers c_i given by Lemma 10 are also constant. Correcting an earlier version of this paper, it was pointed out to me by Professor Lazer (private communication) that constancy of d_i does not ensure constancy of c_i when B depends on t, so (b) must be added as a separate hypothesis. This matter is discussed next.

The fact that B depends continuously on t has nothing to do with the subject, so we assume a set of matrices $\{B_\alpha\}$, where α ranges over an index set I. The letters C, D, K denote positive diagonal matrices which are independent of α unless provided with α as a subscript. A *Volterra multiplier* is a positive diagonal matrix $K = \operatorname{diag} k_i$ such that, in the sense of quadratic forms, $KB > 0$. This means

$$\sum_{i,j} k_i b_{ij} \xi_i \xi_j > 0, \qquad \xi \neq 0, \quad \xi \in \mathbb{R}^n.$$

The class of all matrices admitting a Volterra multiplier is denoted by M_v. In the following examples $I = \{1, 2\}$.

EXAMPLE 1. Here are two matrices that are both row diagonally dominant, yet CB_1 and CB_2 are not both column diagonally dominant for any C:

$$B_1 = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}.$$

EXAMPLE 2. Here are two matrices that satisfy $B_1 > 0$, $B_2 > 0$, yet CB_1 and CB_2 are not both column diagonally dominant for any C, nor are $B_1 D$ and $B_2 D$ both row diagonally dominant for any D:

$$B_1 = \begin{pmatrix} 4 & 1 \\ 6 & 4 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 4 & 6 \\ 1 & 4 \end{pmatrix}.$$

EXAMPLE 3. Here are two matrices that are both row diagonally dominant, yet if ε is sufficiently small they admit no common Volterra multiplier K:

$$B_1 = \begin{pmatrix} 1+\varepsilon & 1 \\ 2 & 2+\varepsilon \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 2+\varepsilon & 2 \\ 1 & 1+\varepsilon \end{pmatrix}, \qquad \varepsilon > 0.$$

In view of the following remark, Example 3 cannot be sharpened to allow both a row and a column multiplier:

Remark 9. Let $\{B_\alpha\}$, $\alpha \in I$ be a set of matrices admitting multipliers C_α and D_α such that $C_\alpha B_\alpha$ is column diagonally dominant and $B_\alpha D_\alpha$ is row diagonally dominant for $\alpha \in I$. Suppose further that these multipliers can be so normalized that $K = (D_\alpha)^{-1} C_\alpha$ is independent of α. Then K is a common Volterra multiplier for all the matrices B_α.

Proof. For simplicity we write $B = B_\alpha$, $C = C_\alpha$, $D = D_\alpha$. If B is diagonally dominant, $Bx = 0$ implies $x = 0$. This shows that $\det B \neq 0$. In fact, $\det B > 0$, as seen by consideration of

$$\det(B + tI), \qquad 0 \leqslant t < \infty.$$

The same applies to each principal subdeterminant, and the Hurwitz criterion now gives $B > 0$ when B is symmetric.

For the general case, let C, D be multipliers such that BD and CB are respectively row and column diagonally dominant. Then CBD is both row and diagonally dominant, so the same holds for the symmetric matrix

$$S = CDB + (CDB)^{\mathrm{T}}.$$

By what we have already proved, the quadratic form associated with S is positive definite; hence that associated with CBD is also. If $x \in \mathbb{R}^n$ and $y = Dx$, the equation

$$x^{\mathrm{T}} CBDx = (Dx)^{\mathrm{T}} D^{-1} CB(Dx) = y^{\mathrm{T}}(D^{-1}C)\, By$$

holds because $D = D^{\mathrm{T}}$. This shows that $K = D^{-1}C$ is a Volterra multiplier and completes the proof.

When used together with Lemma 10, Remark 9 implies the familiar fact [13] that the class $M_r = M_c$ is contained in M_v. The inclusion is strict, since $I + S \in M_v$ where S is any skew-symmetric matrix. Actually there are a great many conditions that ensure $B \in M_v$, none of which imply $B \in M_r$. See [17, 23].

6. Volterra's Methods Revisited

We will now compare some of the foregoing results with others based on conditions of Volterra type. The comparison gains interest from the fact that Volterra's class M_v is much larger than the class $M_r = M_c$ associated with diagonal dominance.

The correlation with quadratic forms goes more smoothly with the Euclidean norm

$$\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$$

than with the norms used above. Our first result uses Volterra's Liapunov function

$$L = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

where the c_i are positive constants. A set $S \subset \mathbb{R}^n$ is called *invariant* relatively to E if $u(t_0) \in S \Rightarrow u(t) \in S$, $t \geqslant t_0$. Our immediate goal is to prove invariance of the set of positive vectors $u \in \mathbb{R}^n$ for which $L \geqslant \alpha$ or $L \leqslant \beta$, where α and

β are positive constants. With the same c as in the definition of L we will need one or the other of the two conditions

$$\lambda^{+}(t)\,\|c\|\,\|u\|^{2} \leqslant \sum_{i,j=1}^{n} c_i b_i(t)\,u_i u_j \leqslant \mu(t)\,\|c\|\,\|w\|^{2}. \tag{fg}$$

It is assumed that $\lambda^{+}(t)$ and $\mu(t)$ are positive functions and that u satisfies E. The notation λ^{+} emphasizes the fact that the inequalites are needed only for $u>0$.

LEMMA 11. (i) *Let L be defined as above, where u satisfies E.*

(i) *For $t \geqslant t_0$ suppose* (f) *holds and that* $\sup a(t)/\lambda^{+}(t) < \infty$. *Then the set defined by $L \leqslant \beta$ is invariant for all sufficiently large constants β.*

(ii) *For $t \geqslant t_0$ suppose* (g) *holds and that* $\inf a(t)/\mu(t) > 0$. *Then the set defined by $L \geqslant \alpha$ is invariant for all sufficiently small constants >0.*

Proof of (i). Note that

$$\dot{L} = \sum_{i=1}^{n} c_i u_i a_i(t) - \sum_{i,\ j=1}^{n} c_i b_{ij}(t)\,u_i u_j \leqslant \|c\|\,\|u\|\,(a^{+}(t) - \lambda^{+}(t)\,\|u\|)$$

where $a^{+}(t) = \max_i a_i^{+}(t)$. Also $L=\beta \Rightarrow \|c\|\,\|u\| \geqslant \beta$, which gives $\dot{L}<0$ if

$$\beta > \|c\|\,\sup \frac{a^{+}(t)}{\lambda^{+}(t)}.$$

Thus $L=\beta \Rightarrow \dot{L}<0$. Hence $L(t_0) \leqslant \beta \Rightarrow L(t) < \beta$ for $t>t_0$.

Proof of (ii). If $L=\alpha$ then $c_i u_i \leqslant \alpha$, so $\|c\|\,\|u\|^{2} \leqslant c_0 \alpha^{2}$ where c_0 is a positive constant whose value need not concern us. By (g) the equation $L=\alpha$ implies

$$\dot{L} \leqslant (\min_i a_i)\,\alpha - c_0 \mu \alpha^{2}.$$

Dividing by μ and using the hypothesis (ii) we see that this is positive if α is sufficiently small. Thus $L=\alpha \Rightarrow \dot{L}>0$. The invariance follows from this.

Another Liapunov-type function that has been used in connection with autonomous systems is easily extended to the general case. Here too, the basic idea goes back to Volterra. We replace both inequalities (fg) by a corresponding inequality

$$\sum_{i,j=1}^{n} c_i b_{ij}(t)\,w_i w_j \geqslant \lambda(t)\,\|c\|\,\|w\|^{2} \tag{h}$$

in which $\lambda(t)$ is positive but w_i can be negative. Hence in general $\lambda^+ > \lambda$. Since (h) ensures that $B(t)$ is nonsingular, we can define functions $q_i(t)$ by

$$\sum_{j=1}^{n} b_{ij}(t)\, q_j(t) = a_i(t). \tag{10}$$

We now make the standard substitutions

$$u_i(t) = q_i e^{v_i(t)}, \qquad w_i(t) = u_i(t) - q_i = q_i(e^{v_i(t)} - 1)$$

where the q_i are positive constants, and we introduce the standard Liapunov function

$$W(t) = \sum_{i=1}^{n} c_i q_i (e^{v_i(t)} - v_i(t) - 1)$$

with $c_i > 0$. Since $\dot{W} = \sum c_i w_i \dot{v}_i$ and $\dot{u}_i = u_i \dot{v}_i$, a short calculation gives

$$\dot{W} = \sum_{i,j=1}^{n} c_i w_i b_{ij}(t)(q_j(t) - q_j) - \sum_{i,j=1}^{n} a_i b_{ij}(t)\, w_i w_j.$$

Hence

$$\dot{W} \leqslant \sum_i c_i \kappa_i(t)\, w_i - \lambda(t)\, \|a\|\, \|w\|^2 \tag{11}$$

where

$$\kappa_i(t) = \sum_{j=1}^{n} b_{ij}(t)(q_j(t) - q_j). \tag{12}$$

The size of $\kappa_i(t)$ depends on how closely $q_j(t)$ can be approximated by constants q_j. With $\kappa(t) = \max_i |\kappa_i(t)|$, Eqs. (12) give

$$\dot{W} \leqslant \|a\|\, \|w\|\, (\kappa(t) - \lambda(t)\, \|w\|).$$

If $\|v\|$ is large then $\|w\|$ is large or some w_i is close to q_i. These results lead to:

Lemma 12. *Let u satisfy E where* (h) *holds. With* $\kappa(t) = \max_i |\kappa_i(t)|$, *suppose* $\sup |\kappa(t)|/\lambda(t) < \min_j q_j$. *Then the set in which $W(t) \leqslant \beta$ is invariant for large β.*

The sets $L \leqslant \beta$, $L \geqslant \alpha$ and $W \leqslant \beta$ in Lemmas 11 and 12 are convex, so consideration of the Poincaré map yields:

THEOREM 2. *Suppose* $a(t)$ *and* $B(t)$ *both have period* $T>0$ *and that one of the following holds*:

(i) $a(t)>0$ *and* $\lambda^+(t)>0$ *for* $0\leqslant t\leqslant T$, *or*

(ii) $\lambda(t)>0$ *and* $\kappa(t)/\lambda(t)<\min_i q_i$ *for* $0\leqslant t\leqslant T$.

Then E *has a nonzero periodic solution.*

7. SUPPLEMENTARY REMARKS

As explained in [17, 20, 22, 23], a constant matrix B belongs to the Volterra class A_0 if there is a multiplier C such that, in the sense of quadratic forms, $CB\geqslant 0$. (Thus $A_0=M_v$.) It belongs to the class A if $\tilde{B}\in A_0$ whenever $\tilde{B}$ is obtained from B by a sufficiently small perturbation of its nonzero elements. The condition $B\in A$ implies $b_{ii}\geqslant 0$ but not $b_{ii}>0$. Associated with B is the *reduced graph* $R(B)$, which can be of type $\odot$, $\oplus$ or $\bullet$. Details are given in [20, 22]. Suffice it to say here that $R(B)$ is uniquely determined by B and can be found by a systematic procedure involving little calculation.

Remark 10. Let $\lim_{t\to\infty} a(t)=a$ and $\lim_{t\to\infty} B(t)=B$ where $B\in A$ and $R(B)$ is of type $\bullet$. Suppose further that $Bq=a$ has a solution $q>0$. Then every solution of E satisfies $\lim_{t\to\infty} u(t)=q$.

Proof. By [20] every solution of the limiting autonomous system tends to q, so the statement follows from the classical theorems of Markus [12].

Remark 10 extends to broad classes of equations in which $\lim B(t)=B$ is not in the class A, provided the limiting equation forms a *chain* in the sense [18]. If $R(B)$ is of Type $\oplus$, then every solution of the autonomous equation has a limit, but the limit depends on the initial condition. The problem of extending Remark 10 to that case is left open, though recent results of Thieme [26, 27, 28] suggest that a suitable extension exists if the passage to the limit is not too slow.

Next let us consider E with $t_0=0$ under the hypothesis that $a=a(t)$ but that B is independent of t. This is the case, for example, in Theorem 1 (ix). Assuming existence of the relevant limits, let

$$\bar{a}=\lim_{T\to\infty}\frac{1}{T}\int_0^T a(t)\,dt,\qquad \bar{u}=\lim_{T\to\infty}\frac{1}{T}\int_0^T u(t)\,dt,\qquad c=\lim_{t\to\infty}\frac{\log u(t)}{t},$$

where $\log u=(\log u_1, \log u_2, ..., \log u_n)$. Integrating $\dot{u}_i/u_i$ yields

$$c=\bar{a}-B\bar{u}. \tag{13}$$

The following properties are probably well known and in any case are easily verified:

Remark 11. Existence of c and $\bar{a}$ implies $c_i \bar{u}_i = 0$; if u is weakly permanent then $c = 0$; and if B is nonsingular then existence of any two of $\bar{a}$, $\bar{u}$, c implies existence of all three.

Assuming u weakly permanent, suppose $\bar{a}$ exists. Then $\bar{u}$ exists by Remark 11, and the equation $B\bar{u} = \bar{a}$ from (13) gives a positive stationary point $q = \bar{u}$ for the autonomous system obtained when $a(t)$ is replaced by $\bar{a}$. If we use this for q in Lemma 12, the two equations

$$Bq = \bar{a}, \qquad Bq(t) = a(t)$$

yield $\kappa(t) = a(t) - \bar{a}$. Hence the main hypothesis of Lemma 12 reduces to

$$|a_i(t) - \bar{a}_i| < \lambda q_j, \qquad i, j = 1, 2, ..., n,$$

where $q = B^{-1}\bar{a}$. Thus the conclusion holds if $a(t)$ is sufficiently well approximated by its mean value $\bar{a}$. This applies in particular to Theorem 2, since $\bar{a}$ exists automatically when a is periodic.

In general one would not expect Equation E with variable coefficients to have a stationary solution $u = q > 0$. However there is an interesting situation in which this always happens:

Remark 12. If u and v both satisfy E, the ratios $w_i = v_i / u_i$ satisfy

$$\frac{\dot{w}_i}{w_i} = \bar{a}_i - \sum_{j=1}^{n} b_{ij} u_j w_j \qquad \text{where} \quad \bar{a}_i = \sum_{j=1}^{n} b_{ij} u_j.$$

Hence w_i are members of a Lotka–Volterra population with stationary point $w_i = 1$ and coupling matrix $(b_{ij} u_j)$. We omit the trivial proof.

In conclusion, we explain why it was not possible just to revise the A-L proofs to allow $b_{ij} < 0$. On a given interval let

$$m(t) = \max_i |u_i(t) - v_i(t)|$$

and let $I(t)$ be the set of indices $p \in \{1, ..., n\}$ for which

$$m(t) = |u_p(t) - w_p(t)|.$$

The main results [4] depend on the novel and ingenious Liapunov function

$$\phi(t) = \max_{p \in I(t)} |\log u_p(t) - \log v_p(t)|.$$

Ahmad and Lazar prove that the upper right Dini derivate satisfies

$$D^+\phi(t) = \limsup_{h \to 0+} \frac{\phi(t+h) - \phi(t)}{h} \leqslant -\delta(t)\, m(t).$$

From this they draw a conclusion that in the case $\delta(t) \geqslant 0$ would imply that ϕ is nonincreasing. However, their proof assumes that ϕ is continuous, which is rarely the case. To be sure, the Zygmund conditions [16] allow discontinuities in theorems of this kind, but instead of these conditions ϕ satisfies

$$\phi(t) \geqslant \limsup_{h \to 0} \phi(t+h). \tag{14}$$

This is insufficient. For example $\phi(t) = [t] - t$ satisfies (14) and $D^+\phi = -1$, yet is not monotone.

Acknowledgments

As explained at appropriate points above, Part I of this paper owes much to [4] and in particular to Professor Lazer. I am indebted to Professor Basil Gordon for helpful conversations in the initial stages of both Parts I and II. It is also a pleasure to acknowledge the exceptionally competent and careful report of the Referee, which led to several improvements in the exposition.

References

1. S. Ahmad, On almost periodic solutions of the competing species problems, *Proc. Amer. Math. Soc.* **102** (1988), 855–861.
2. S. Ahmad, Convergence and ultimate bounds of solutions of the nonautonomous Volterra–Lotka competition equations, *J. Math. Anal. Appl.* **127** (1987), 377–387.
3. S. Ahmad and A. C. Lazer, Asymptotic behavior of solutions of periodic competition diffusion system, *Nonlin. Anal., Theory, Methods and Applications* **13** (1989), 263–284.
4. S. Ahmad and A. C. Lazer, On the nonautonomous *N*-competing species problems, preprint 1992.
5. C. Alvarez and A. C. Lazer, An application of topological degree to the periodic competing species problem, *J. Austral. Math. Soc. Series B* **28** (1986) 202–219.
6. L. Amerio, Soluzioni quasi-periodiche, o limitate, di sistemi differenziali non lineari quasi-periodici, o limitati, *Ann. Mat. Pura Appl.* **39** (1955), 97–119.
7. H. Bohr, Almost Periodic Functions, Chelsea (1951), copyright 1933 by Julius Springer, Berlin, 36–38.
8. K. Gopalsamy, Global asymptotic stability in a periodic Lotka–Volterra system, *J. Austral. Math. Soc. Series B* **27** (1986) 66–72.
9. K. Gopalsamy, Global asymptotic stability in an almost periodic Lotka–Volterra system, ibid., 346–360.

10. J. Hofbauer and K. Sigmund, The Theory of Evolution and Dynamical Systems, Cambridge University Press.
11. D. O. Logofet, Matrices and Graphs, CRC Press (1992).
12. L. Markus, Asymptotically autonomous differential systems, Contributions to the Theory of Nonlinear Oscillations III, *Ann. Math. Study* **36**, Princeton University Press (1956), 17–29.
13. P. J. Moylan, Matrices with positive principal minors, *Linear Alg. Apps.* **17** (1977), 53–58.
14. C. F. Muckenhoupt, Almost periodic functions and vibrating systems, *J. Math. Phys. Mass. Inst. Tech.* **8** (1929), 163–200.
15. H. Nikaido, "Convex Structures and Economic Theory," Academic Press (1968).
16. R. Redheffer, Increasing functions, *Aequat. Math.* **22** (1981), 119–133.
17. R. Redheffer, Volterra multipliers I, II, *SIAM J. Alg. Discrete Methods* **6**, (1985), 592–623. [Part I contains a bibliography of 41 references.]
18. R. Redheffer, A new class of Volterra differential equations for which the solutions are globally asymptotically stable, *J. Differential Equations* **82** (1989), 251–268.
19. R. Redheffer, R. Redlinger, and W. Walter, A theorem of LaSalle–Lyapunov type for parabolic systems, *SIAM J. Math. Anal.* **19** (1988), 121–132.
20. R. Redheffer and W. Walter, Solution of the stability problem for a class of generalized Volterra prey–predator systems, *J. Differential Equations* **52** (1984), 145–263.
21. R. Redheffer and W. Walter, On parabolic systems of the Volterra prey–predator type, *J. Nonlinear Anal.* **7** (1983), 333–347.
22. R. Redheffer and Z. Zhou, Global asymptotic stability for a class of many-variable Volterra prey–predator systems, *Nonlinear. Anal. TMA* **5** (1981), 1309–1329.
23. R. Redheffer and Z. Zhou, A class of matrices conected with Volterra prey–predator equations, *SIAM J. Alg. Disc. Meth.* **3** (1982) 122–134.
24. R. Redheffer and Z. Zhou, Sign semistability and global asymptotic stability, *Ann. Differential Equations* **5** (1989) 145–153.
25. L. Scheeffer, Zur Theorie der stetigen Funktionen einer reelen Veränderlichen, *Acta Math.* **5** (1884/1885), 279–296.
26. H. R. Thieme, Convergence results and a Poincaré–Bendixson trichotomy for asymptotically autonomous differential equations, *J. Math. Biol.* **30** (1992), 755–763. [This contains a bibliography of 30 references.]
27. H. R. Thieme, Asymptotically autonomous differential equations in the plane, manuscript.
28. H. R. Thieme, Asymptotically autonomous differential equations in the plane, II. Stricter Poincaré–Bendixson type results, manuscript.
29. A. Tineo, On the asymptotic behavior of some population models, *J. Math. Anal. Appl.* **169** (1992), 516–529.
30. A. Tineo, An iterative scheme for the *N*-competing species problem, manuscript.
31. A. Tineo and C. Alvarez, A different consideration about the globally asymptotically stable solution of the periodic *n*-species problem, *J. Math. Anal. Appl.* **159** (1991), 44–50.
32. V. Volterra, "Leçons sur la théorie mathématique de la lutte pour la vie," Gauthier–Villars, Paris, 1931.
33. N. Wiener, Generalized harmonic analysis, *Acta Math.* **55** (1930), 117–258.